Exercise 4

Evaluate the line integral, where C is the given curve.

$$\int_C xe^y\,ds, \quad C \text{ is the line segment from } (2,0) \text{ to } (5,4)$$

Solution

The line going through $(2,0)$ and $(5,4)$ is

$$y = \frac{4}{3}x - \frac{8}{3}.$$

Parameterize it by setting $x = t$, which then means $y = (4/3)t - 8/3$, and having $2 \le t \le 5$. With this parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C xe^y\,ds &= \int_2^5 x(t)e^{y(t)}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_2^5 t\exp\left(\frac{4}{3}t - \frac{8}{3}\right)\sqrt{(1)^2 + \left(\frac{4}{3}\right)^2}\,dt \\
&= \int_2^5 te^{4t/3}e^{-8/3}\sqrt{\frac{25}{9}}\,dt \\
&= \frac{5}{3}e^{-8/3}\int_2^5 te^{4t/3}\,dt \\
&= \frac{5}{3}e^{-8/3}\int_2^5 \frac{\partial}{\partial a}(e^{at})\bigg|_{a=4/3}\,dt \\
&= \frac{5}{3}e^{-8/3}\frac{d}{da}\left(\int_2^5 e^{at}\,dt\right)\bigg|_{a=4/3} \\
&= \frac{5}{3}e^{-8/3}\frac{d}{da}\left(\frac{1}{a}e^{at}\bigg|_2^5\right)\bigg|_{a=4/3} \\
&= \frac{5}{3}e^{-8/3}\frac{d}{da}\left(\frac{e^{5a} - e^{2a}}{a}\right)\bigg|_{a=4/3} \\
&= \frac{5}{3}e^{-8/3}\left[\frac{(5e^{5a} - 2e^{2a})a - (e^{5a} - e^{2a})}{a^2}\right]\bigg|_{a=4/3} \\
&= \frac{5}{3}e^{-8/3}\left[\frac{(5a-1)e^{5a} + (1-2a)e^{2a}}{a^2}\right]\bigg|_{a=4/3} \\
&= \frac{5}{3}e^{-8/3}\left[\frac{\left(\frac{20}{3} - 1\right)e^{20/3} + \left(1 - \frac{8}{3}\right)e^{8/3}}{\left(\frac{4}{3}\right)^2}\right] = \frac{5}{16}(17e^4 - 5).
\end{aligned}$$